CORRELATION FUNCTIONS OF AN ELASTIC FIELD
OF QUASI-ISOTROPIC SOLID BODIES

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The strain of inhomogeneous materials is usually accompanied by distortion in the form of the component grains. These distortions are random and, even if the average value of the strains does not exceed the value of the elastic limit, local stresses may be large enough to cause microplastic shears. It is the latter phenomenon that prompts the study of not only the average characteristics of an elastic field, but also the deviations of the strain field from its mean value in the isolated regions of a solid body.

We find that various correlation functions may be used to study the relationship between the regions of inhomogeneity. These functions make it possible to construct quantitative pictures of microstresses, of deviations of the grains from their equilibrium form and of their interaction during the strain process.

We shall assume the second order correlation function for the elastic moduli tensor known, and we shall choose it in the form corresponding to an inhomogeneous medium with the boundaries between the grains clearly defined. Distortion of the grains during the strain process are characterized by the binary correlation functions of the stress and strain tensors and angles of rotation. To obtain the information on the character of the deformation of grains, we analyze the tensor and coordinate relationships of the correlation functions indicated above.

1. Let us obtain the autocorrelation tensor of elastic moduli of randomly oriented polycrystalline specimen. We shall express the elastic moduli tensor λ_{ijkl} in an arbitrary coordinate system in the terms of its value in the crystallographic reference system $\overset{\circ}{\lambda}_{pqrs}$ and the direction cosines α_{ip}

$$\lambda_{ijkl} = \alpha_{ip}\alpha_{jq}\alpha_{kr}\alpha_{ls}\overset{\circ}{\lambda}_{pqrs} \qquad (1.1)$$

Then the problem of computing the second order central moment function of the elastic moduli tensor is reduced to the process of averaging the products of direction cosines

$$A_{pqrs}^{ijkl} \equiv \langle \lambda'_{ijkl}\lambda'_{pqrs} \rangle = [\langle \alpha_{ia}\alpha_{jb}\alpha_{kc}\alpha_{ld}\alpha_{pt}\alpha_{qu}\alpha_{rv}\alpha_{sw} \rangle -$$
$$- \langle \alpha_{ia}\alpha_{jb}\alpha_{kc}\alpha_{ld} \rangle \langle \alpha_{pt}\alpha_{qu}\alpha_{rv}\alpha_{sw} \rangle] \overset{\circ}{\lambda}_{abcd}\overset{\circ}{\lambda}_{tuvw} \qquad (1.2)$$

Here the brackets $\langle \ \rangle$ denote the averaging over all possible orientations of the crystallites, while the primes denote the random terms of the elastic moduli tensor

$$\lambda'_{ijkl} = \lambda_{ijkl} - \langle \lambda_{ijkl} \rangle$$

Direct computation yields $\langle \alpha_{im} \alpha_{jn} \rangle = {}^{1}\!/_{3} \delta_{ij} \delta_{mn}$

$$\langle \alpha_{im} \alpha_{jm} \alpha_{kn} \alpha_{ln} \rangle = \frac{1}{5!!} [\delta_{ijkl} + {}^{1}\!/_{2} (\delta_{mn} - 1)(3\delta_{ijkl} - 5\delta_{ij}\delta_{kl})] \qquad (1.3)$$

$$\langle \alpha_{im} \alpha_{jm} \alpha_{km} \alpha_{pn} \alpha_{qn} \alpha_{rn} \rangle = \frac{1}{7!!} \delta_{ijkpqr} \delta_{mn}$$

$$\langle \alpha_{im} \alpha_{jm} \alpha_{km} \alpha_{lm} \alpha_{pn} \alpha_{qn} \rangle = \frac{1}{7!!} [\delta_{ijklpq} \delta_{mn} + {}^{1}\!/_{2}(\delta_{mn} - 1)(3\delta_{ijklpq} - 7\delta_{ijkl}\delta_{pq})]$$

$$\langle \alpha_{im} \alpha_{jm} \alpha_{km} \alpha_{lm} \alpha_{pvn} \alpha_{qn} \alpha_{rn} \alpha_{sn} \rangle =$$
$$= \frac{1}{9!!} [\delta^{ijkl}_{pqrs} + \frac{1}{8}(\delta_{mn} - 1)(5\delta^{ijkl}_{pqrs} - 63\delta_{ijkl}\delta_{pqrs} + 9\beta^{ijkl}_{pqrs})]$$

where $\delta_{ij...rs}$ denotes the sum of products of the Kronecker deltas and where the summation is performed over all possible permutations of indices except the identity permutations of the type $\delta_{ij} = \delta_{ji}$. The fourth, sixth and eighth order matrices δ contain 3, 15 and 105 terms respectively

$$\delta_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \qquad (1.4)$$
$$\delta_{ijklpq} = \delta_{ij}\delta_{klpq} + \delta_{ik}\delta_{jlpq} + \delta_{il}\delta_{jkpq} + \delta_{ip}\delta_{jklq} + \delta_{iq}\delta_{jklp}$$
$$\delta^{ijkl}_{pqrs} = \delta_{ij}\delta_{klpqrs} + \delta_{ik}\delta_{jlpqrs} + \delta_{il}\delta_{jkpqrs} + \delta_{ip}\delta_{jklqrs} +$$
$$+ \delta_{iq}\delta_{jklprs} + \delta_{ir}\delta_{jklpqs} + \delta_{is}\delta_{jklpqr}$$

In the expression (1.3), β^{ijkl}_{pqrs} denotes the matrix

$$\beta^{ijkl}_{pqrs} = \beta^{pqrs}_{ijkl} = 6\delta_{(ij}\delta_{kl)pqrs} = \delta_{ij}\delta_{klpqrs} + \delta_{ik}\delta_{jlpqrs} + \delta_{il}\delta_{jkpqrs} +$$
$$+ \delta_{jk}\delta_{ilpqrs} + \delta_{jl}\delta_{ikpqrs} + \delta_{kl}\delta_{ijpqrs} \qquad (1.5)$$

When computing the averaged products of the direction cosines (1.3), we have used the transfer matrix constructed with the help of the Euler angles. When $m = n$ the averaging of an even number of the direction cosines leads to the terms of the type

$$\frac{1}{(2n+1)!!} \delta_{ij...s}$$

while the odd number yields zero. When $m \neq n$, the components of the averaged matrix of direction cosines will differ from zero only when all rows and columns contain an even number of the elements α_{in}.

Let us find the autocorrelation tensor A^{ijkl}_{pqrs} for an orthorhombic polycristalline system. The elastic moduli tensor of the cristallite possessing orthorhombic symmetry referred to the crystallographic axes, can be written as follows: $\qquad (1.6)$

$$\overset{\circ}{\lambda}_{ijkl} = \sum_{n=1}^{3} [\lambda^{(n)}\delta_{in}\delta_{jn}\delta_{kn}\delta_{ln} + \mu^{(n)}(\delta_{in}\delta_{jn}\delta_{kl} + \delta_{ij}\delta_{kn}\delta_{ln}) + 4\nu^{(n)}\delta_{n(i}\delta_{j)(k}\delta_{l)n}]$$

where the following symmetrization is performed over the indices within the round brackets $\delta_{n(i}\delta_{j)k} = {}^{1}\!/_{2}(\delta_{ni}\delta_{jk} + \delta_{nj}\delta_{ik})$

Coefficients $\lambda^{(n)}$, $\mu^{(n)}$ and $\nu^{(n)}$ are related to the matrix stress constants as follows:

$$\lambda^{(1)} = c_{11} + c_{23} + 2c_{44} - (c_{12} + c_{13} + 2c_{55} + 2c_{66})$$
$$\lambda^{(2)} = c_{22} + c_{13} + 2c_{55} - (c_{12} + c_{23} + 2c_{44} + 2c_{66})$$
$$\lambda^{(3)} = c_{33} + c_{12} + 2c_{66} - (c_{13} + c_{23} + 2c_{44} + 2c_{55})$$
$$2\mu^{(1)} = c_{12} + c_{13} - c_{23}, \qquad 2\mu^{(2)} = c_{12} + c_{23} - c_{13}$$
$$2\mu^{(3)} = c_{13} + c_{23} - c_{12}, \qquad 2\nu^{(1)} = c_{55} + c_{66} - c_{44} \qquad (1.7)$$
$$2\nu^{(2)} = c_{44} + c_{66} - c_{55}, \qquad 2\nu^{(3)} = c_{44} + c_{55} - c_{66}$$

Inserting (1.6) into (1.2) and using Formulas (1.3), we find the autocorrelation tensor of elastic moduli for orthorhombic polycrystals

$$A_{pqrs}^{ijkl} = \frac{1}{9\text{II}} \Big[\frac{1}{40} \Big(\sum_n \lambda^{(n)} \lambda^{(n)} - \sum_{n,m} \lambda^{(n)} \lambda^{(m)} \Big) \times \tag{1.8}$$

$$\times (45\beta_{pqrs}^{ijkl} - 35\delta_{pqrs}^{ijkl} - 63\delta_{ijkl}\delta_{pqrs}) + P_{\lambda\lambda} \Big(\delta_{pqrs}^{ijkl} - \frac{12}{5} \delta_{ijkl}\delta_{pqrs} \Big) +$$

$$+ \frac{2}{7\text{II}} P_{\lambda\mu} \gamma_{pqrs}^{ijkl} + \frac{2}{7\text{II}} P_{\lambda\nu} \Big(\beta_{pqrs}^{ijkl} - \frac{14}{3} \delta_{ijkl}\delta_{pqrs} - \gamma_{pqrs}^{ijkl} \Big) + \frac{4}{5\text{II}} P_{\mu\mu} \delta_{[ij} D_{kl][pq}\delta_{rs]} +$$

$$+ \frac{12}{5\text{II}} P_{\mu\nu} \Big(\delta_{[ij} D_{kl]}{}_{(pq}\delta_{rs)} + \delta_{(ij} D_{kl)}{}_{[pq}\delta_{rs]} - \frac{2}{3} \delta_{r[ij} D_{kl]}{}_{[pq}\delta_{rs]} \Big) +$$

$$+ \frac{4}{5\text{II}} P_{\nu\nu} \big(9\delta_{(ij} D_{kl)}{}_{(pq}\delta_{rs)} - 3\delta_{(ij} D_{kl)}{}_{[pq}\delta_{rs]} - 3\delta_{[ij} D_{kl]}{}_{(pq}\delta_{rs)} + \delta_{[ij} D_{kl]}{}_{[pq}\delta_{rs]} \big)$$

where

$$P_{\lambda\mu} \equiv \frac{1}{2} \Big(3 \sum_n \lambda^{(n)} \mu^{(n)} - \sum_{n,m} \lambda^{(n)} \mu^{(m)} \Big) \tag{1.9}$$

$$\gamma_{pqrs}^{ijkl} \equiv \delta_{ijkl[pq}\delta_{rs]} + \delta_{pqrs[ij}\delta_{kl]} - \frac{7}{3} \big(\delta_{ijkl}\delta_{pq}\delta_{rs} + \delta_{pqrs}\delta_{ij}\delta_{kl} \big)$$

$$D_{ijkl} \equiv 2 \big(\delta_{i(k}\delta_{l)j} - \frac{1}{3}\delta_{ij}\delta_{kl} \big)$$

and where the symmetrization $\delta_{ijkl[pq}\delta_{rs]} \equiv \frac{1}{2} \big(\delta_{ijkl}\delta_{pq}\delta_{rs} + \delta_{ijkl}\delta_{rs}\delta_{pq} \big)$

is performed over the indices within square brackets. Expression (1.8) yields the contractions of the autocorrelation tensor

$$A_{klqq}^{ijpp} = \frac{1}{10} A_1 D_{ijkl}, \quad D_{pqrs} A_{klrs}^{ijpq} = \frac{1}{40} (10 A_1 \delta_{ij}\delta_{kl} + 3 A_2 D_{ijkl}) \tag{1.10}$$

$$A_1 = \frac{2}{3} P_{\varkappa\varkappa}, \quad A_2 = \frac{2}{3} P_{\xi\xi} + 8 P_{\nu\nu} + \frac{2}{5} \sum_{n,m} \lambda^{(n)} \lambda_j^{(m)}$$

$$\varkappa^{(n)} = \xi^{(n)} + 3\mu^{(n)}, \quad \xi^{(n)} = \lambda^{(n)} + 4\nu^{(n)}$$

When the polycrystal exhibits higher symmetry, Formulas (1.6) − (1.10) simplify, and the elastic moduli tensor becomes

$$\lambda_{ijkl}^t = \lambda_{ijkl}^h + \lambda_6 \sum_n^3 \delta_{in}\delta_{jn}\delta_{kn}\delta_{ln} \tag{1.11}$$

for the tetragonal system [1],

$$\lambda_{ijkl}^h = \lambda_{ijkl}^{is} + \lambda_3 \delta_{i3}\delta_{j3}\delta_{k3}\delta_{l3} + \lambda_4 (\delta_{ij}\delta_{k3}\delta_{l3} + \delta_{i3}\delta_{j3}\delta_{kl}) + 4\lambda_5 \delta_{3(i}\delta_{j)(k}\delta_{l)3} \tag{1.12}$$

for the hexagonal system,

$$\lambda_{ijkl}^c = \lambda_{ijkl}^{is} + \lambda_6 \sum_n^3 \delta_{in}\delta_{jn}\delta_{kn}\delta_{ln} \tag{1.13}$$

for the cubic system and

$$\lambda_{ijkl}^{is} = \lambda_1 \delta_{ij}\delta_{kl} + 2\lambda_2 \delta_{i(k}\delta_{l)j} \tag{1.14}$$

for the isotropic system.

In the case of tetragonal symmetry we have

$$c_{11} = c_{22}, \quad c_{13} = c_{23}, \quad c_{44} = c_{55}$$

which leads to the following relations between the one-index coefficients $\lambda^{(n)}$, $\mu^{(n)}$, $\nu^{(n)}$ and λ_i:

$$\lambda^{(1)} = \lambda^{(2)} = \lambda_6, \quad \lambda^{(3)} = \lambda_3 + \lambda_6, \quad \mu^{(1)} = \mu^{(2)} = \frac{1}{2}\lambda_1 \tag{1.15}$$

$$\mu^{(3)} = \lambda_4 + \frac{1}{2}\lambda_1, \quad \nu^{(1)} = \nu^{(2)} = \frac{1}{2}\lambda_2, \quad \nu^{(3)} = \lambda_5 + \frac{1}{2}\lambda_2$$

This yields λ_l in terms of c_{lk}

$$\lambda_1 = c_{13}, \qquad \lambda_2 = c_{66}, \qquad \lambda_3 = c_{33} + 2c_{13} + 4c_{66} - c_{11} - 2c_{13} - 4c_{44} \quad (1.16)$$
$$\lambda_4 = c_{13} - c_{13}, \qquad \lambda_5 = c_{44} - c_{66}, \qquad \lambda_6 = c_{11} - c_{13} - 2c_{66}$$

When $\lambda_6 = 0$ the relations (1.16) lead to the hexagonal symmetry, while when $\lambda_3 = \lambda_4 = \lambda_5 = 0$ we have the cubic symmetry. In the latter case the autocorrelation tensor (1.8) becomes very simple

$$A^{ijkl}_{pqrs} = \frac{\lambda_6^2}{60 \cdot 7 \, \text{II}} \left(35\delta^{ijkl}_{pqrs} + 63\delta_{ijkl}\delta_{pqrs} - 45\beta^{ijkl}_{pqrs} \right) \tag{1.17}$$

Coefficients A_1 and A_2 for the tetragonal and hexagonal system are, respectively

$$A^t_1 = {}^2/_3 (\lambda_3 + 3\lambda_4 + 4\lambda_5)^2$$
$$A^t_2 = {}^2/_3 (\lambda_3 + 4\lambda_5)^2 + {}^3/_5 (\lambda_3 + 3\lambda_6)^2 + 8\lambda_5^2$$
$$A^h_1 = A^t_1, \qquad A^h_2 = {}^4/_{15} (2\lambda_3 + 5\lambda_5)^2 + 12\lambda_3^2$$

while for the cubic system they become

$$A^c_1 = 0, \qquad A^c_2 = {}^{18}/_5 \lambda_6^2 \tag{1.18}$$

Lomakin [3] obtained the following autocorrelation tensor for an arbitrary, quasi-homogeneous, randomly oriented medium

$$A^{ijkl}_{pqrs} = h_1 \delta_{ij}\delta_{kl}\delta_{pq}\delta_{rs} + 2h_2 (\delta_{ij}\delta_{kl}\delta_{p(r}\delta_{s)q} + \delta_{i(k}\delta_{l)j}\delta_{pq}\delta_{rs}) + 4h_3 \delta_{i(k}\delta_{l)j}\delta_{p(r}\delta_{s)q} \quad (1.19)$$

where h_l are constants. We see that the autocorrelation tensor given by (1.17) contains, in the simplest cubic symmetry case, 105 different terms, while the expression (1.19) contains only nine terms.

The unsuitability of (1.19) for the present case is shown on the following example. Direct computation of a component of the autocorrelation tensor A^{1231}_{1231} yields

$$A^{1231}_{1231} = \langle \lambda'^2_{1231} \rangle = \lambda_6^2 \left\langle \left(\sum_n \alpha_{1n}^2 \alpha_{2n}\alpha_{3n} - \frac{1}{15}\delta_{1231} \right)^2 \right\rangle = \lambda_6^2 \sum_{n,m} \langle \alpha_{1n}^2 \alpha_{1m}^2 \alpha_{2n}\alpha_{2m}\alpha_{3n}\alpha_{3m} \rangle =$$
$$= 3\lambda_6^2 \langle \alpha_{1n}^4 \alpha_{2n}^2 \alpha_{3n}^2 \rangle = \frac{1}{7 \text{II}} \lambda_6^2 \tag{1.20}$$

This can also be obtained from (1.17), while according to (1.19), $A^{1231}_{1231} = 0$. Thus we see that (1.19) can only be used for randomly oriented mechanical mixtures of isotropic components.

In the case of a two-component mixture the coefficients h_l are [4]

$$h_1 = c_1 c_2 (\lambda_1 - \lambda_2)^2, \quad h_2 = c_1 c_2 (\lambda_1 - \lambda_2)(\mu_1 - \mu_2), \quad h_3 = c_1 c_2 (\mu_1 - \mu_2)^2 \quad (1.21)$$

where c_l denotes the concentration, while λ_l and μ_l are the Lamé constants of the lth component. Contractions of the elastic moduli autocorrelation tensor are obtained for the mechanical mixtures of isotropic components, from the relations (1.19) and (1.21)

$$A^{ijpp}_{klqq} = 9D_K \delta_{ij}\delta_{kl}, \qquad D_{pqrs}A^{ijpq}_{klrs} = 4D_\mu D_{ijkl} \tag{1.22}$$

$$K = \lambda + {}^2/_3 \mu, \qquad D_K \equiv \langle K'^2 \rangle, \qquad D_\mu \equiv \langle \mu'^2 \rangle$$

Comparing (1.10) and (1.22) we note, that the contractions A^{ijpp}_{klqq} are, from the point of view of their tensor properties, of opposite character for the polycrystals and for mechanical mixtures.

2. We shall now describe the deviations of the stress and strain fields σ_{ij} and ε_{ij} from their mean values, in the terms of the corresponding correlation functions. The simplest correlation function describing the inhomogeneity of the strain field has the form of a

second rank tensor U_{ij}, which can be expressed in the terms of random components of the displacement field as

$$U_{ij}(\mathbf{r} - \mathbf{\rho}) = \langle u_i'(\mathbf{r}) u_j'(\mathbf{\rho}) \rangle \tag{2.1}$$

Here the angle brackets denote the averaging over a region which is large compared with the spatial dimension of the correlations, but small compared with the intervals over which the regular parts of the function vary substantially.

In addition to U_{ij}, we shall consider three other correlation functions

$$E_{ijmn}(\mathbf{r} - \mathbf{\rho}) = \langle \varepsilon_{ij}'(\mathbf{r}) \varepsilon_{mn}(\mathbf{\rho}) \rangle, \qquad \Omega_{ij}(\mathbf{r} - \mathbf{\rho}) = \langle \omega_i'(\mathbf{r}) \omega_j'(\mathbf{\rho}) \rangle \tag{2.2}$$

$$S_{ijmn}(\mathbf{r} - \mathbf{\rho}) = \langle \sigma_{ij}'(\mathbf{r}) \sigma_{mn}'(\mathbf{\rho}) \rangle, \qquad \omega_i = {}^1/_2 e_{lkj} u_{j,\,k}$$

first of which describes the inhomogeneity of the shear and volume strains, the second describes local rotations, and the third characterizes the inhomogeneity of the stress field. Here e_{lkj} denotes a unit antisymmetric tensor.

We shall use the equilibrium equation

$$L_{il} u_l + f_i = 0, \qquad L_{il} \equiv \nabla_k \lambda_{iklm} \nabla_m = \langle L_{il} \rangle + L_{il}' \tag{2.3}$$

where $\langle L_{il} \rangle$ is the regular and L_{il}' is the random component of the operator L_{il}, to obtain the correlation functions.

From (2.3) it follows, that within the approximation including double correlations, the random component of the displacement vector of a uniformly strained medium can be written as [5]

$$u_l' = G_{ik} * L_{kl}' \langle u_l \rangle = G_{ik,\,l} * \lambda_{klpq} \langle \varepsilon_{pq} \rangle \tag{2.4}$$

where the asterisk denotes the convolution integration over the whole space and G_{ik} is the Green's tensor function of the operator $\langle L_{il} \rangle$. This function can be written in the terms of the averaged Lamé constants $\langle \lambda \rangle$ and $\langle \mu \rangle$

$$G_{ik}(\mathbf{r}) = \frac{1}{8\pi \langle \mu \rangle} (r_{,\,pp} \delta_{ik} - \varkappa r_{,\,ik}), \qquad \varkappa \equiv \frac{\langle \lambda + \mu \rangle}{\langle \lambda + 2\mu \rangle} \tag{2.5}$$

Inserting (2.4) into (2.1) we obtain

$$U_{ij}(\mathbf{r}) = A_{rslv}^{klpq} \langle \varepsilon_{pq} \rangle \langle \varepsilon_{lv} \rangle I_{jrs}^{ikl}(\mathbf{r}) \tag{2.6}$$

$$I_{jrs}^{ikl}(\mathbf{r}) \equiv G_{ik,\,l} * \varphi * G_{jr,\,s}(\mathbf{r}) \tag{2.7}$$

where $\varphi(\mathbf{r})$ is the radial part of the binary correlation function of the elastic moduli tensor

$$\langle \lambda_{ijkl}'(\mathbf{r}) \lambda_{pqrs}'(\mathbf{\rho}) \rangle = A_{pqrs}^{ijkl} \varphi(\mathbf{r} - \mathbf{\rho}) \tag{2.8}$$

Separation of the tensorial from the coordinate relationships for the binary correlation function of the elastic moduli tensor of the quasi-isotropic solid bodies follows from the results of [2 and 6]. Let us carry out the averaging process in two stages, firstly for a single grain, and secondly over all grains in the aggregate. We shall seek the mean value of the random component of the elastic moduli tensor over one grain at the point $\mathbf{\rho}$, if its value at the point $\mathbf{r}$ is known. For this purpose we shall introduce the weight function, with the aid of which the averaging process shall be carried out. This function will be equal to unity if both points, $\mathbf{r}$ and $\mathbf{\rho}$ belong to the same grain, and zero otherwise. If the crystal habit of the crystallites deviates from the crystallographic axes, then the weight function is a scalar. With this in mind we shall draw a sphere of radius $\mathbf{\rho} - \mathbf{r}$ and perform the integration over the angles

$$\overline{\lambda_{ijkl}'(\mathbf{\rho})} = \frac{1}{4\pi} \int \lambda_{ijkl}'(\mathbf{r}) \Phi(|\mathbf{r} - \mathbf{\rho}|, \Omega) d\Omega = \lambda_{ijkl}'(\mathbf{r}) \varphi(|\mathbf{r} - \mathbf{\rho}|) \tag{2.9}$$

This averaging process is performed over the various forms of the crystallites, but not over the orientations of their crystallographic axes. If the function Φ truncates the values of the random components of the elastic moduli tensor on the boundary of the crystallite, then the spherically symmetric function ϕ "blurs" the values of the tensor λ'_{ijkl} over a region comparable in size with the average size of a crystallite. This function is equal to unity when $r = \rho$ and decreases asymptotically with increasing argument.

Let us now perform the averaging over all crystallites. We can assume that the only difference between the various crystallites lies in the orientation of their crystallographic axes and, consequently, perform the averaging over the orientations [2]. Multiplying both sides of (2.9) by $\lambda'_{pqrs}(r)$

$$\overline{\lambda'_{pqrs}(r)\,\lambda'_{ijkl}(\rho)} = \overline{\lambda'_{pqrs}(r)\cdot\lambda'_{ijkl}(r)}\,\phi\,(|\,r - \rho\,|)$$

and averaging over all orientations, we obtain (2.8).

Computation of the integral I^{ikl}_{jrs} gives

$$I^{ikl}_{jrs}(r) = -\frac{1}{\langle\mu\rangle^2}\,[J^{(2)}_{,ls}\delta_{ik}\delta_{jr} + \varkappa\,(J^{(3)}_{,ljrs}\delta_{ik} + J^{(3)}_{,ikls}\delta_{jr}) + \varkappa^2 J^{(4)}_{,ikljrs}\,] \quad (2.10)$$

$$J^{(n)}_{,ik\ldots s}(r) \equiv \nabla_i\nabla_k\ldots\nabla_s\frac{1}{8\pi^3}\int e^{ikr}k^{-2n}\phi^*(k)\,dk \quad (2.11)$$

Functions $J^{(n)}_{,ik\ldots s}$ are given by (2.11) under the condition that the order of differentiation $N \leqslant 2n$, and have the following recurrent relations:

$$\delta_{ik}J^{(n)}_{,ikl\ldots s}(r) = -J^{(n-1)}_{,l\ldots s}(r), \qquad J^{(0)}(r) = \phi(r) \quad (2.12)$$

We shall assume that the inhomogeneity of the material results from the presence of definite boundaries between the separate crystallites. The material properties change abruptly during the passage from one crystallite to the next, and the spatial part of the correlation function can be chosen in the form [4 and 5]

$$\phi(r) = \exp\frac{-r}{a}, \qquad \phi^*(k) \equiv \int e^{-ikr}\phi(r)\,dr = \frac{8\pi a^3}{(1 + a^2 k^2)^2} \quad (2.13)$$

Inserting (2.13) into (2.11) we find the integral $J(r)$

$$J^{(n)}(r) = (-1)^n a^{2n}\left[\left(1 + \frac{2na}{r}\right)\exp\left(-\frac{r}{a}\right) - \frac{2a}{r}\sum_{k=0}^{n-1}\frac{n-k}{(2k)!}\left(\frac{r}{a}\right)^{2k}\right] \quad (2.14)$$

from which we see that the finite function $\phi(r) = \exp(-r/a)$ used to describe the spatial part of the elastic moduli correlation tensor does not lead to the appearance of any singularities in the integrals $J^{(n)}_{,ik\ldots s}$. This is due to the fact that when the integral $J^{(n)}(r)$ is differentiated $2n$ times, all singular derivatives vanish. For example, for $J_{,ij}^{(1)}$ we have

$$J^{(1)} = -a^2\left[\left(1 + \frac{2}{\rho}\right)e^{-\rho} - \frac{2}{\rho}\right] \qquad \left(\rho \equiv \frac{r}{a}\right)$$

$$\nabla_i\nabla_j J_S^{(1)} = 2a^2(1 - e^{-\rho})\nabla_i\nabla_j\left(\frac{1}{\rho}\right)_S = -\frac{8\pi}{3}(1 - e^{-\rho})\delta(\rho)\delta_{ij} \equiv 0$$

$$\nabla_i\nabla_j J_F^{(1)} = T_1^{(1)}\delta_{ij} + T_2^{(1)}\psi_{ij}, \qquad \psi_{ij} = n_i n_j \qquad \left(n_i = \frac{x_i}{r}\right)$$

$$T_1^{(1)} = \left(\frac{1}{\rho} + \frac{2}{\rho^2} + \frac{2}{\rho^3}\right)e^{-\rho} - \frac{2}{\rho^3}, \qquad T_2^{(1)} = -\left(1 + \frac{3}{\rho} + \frac{6}{\rho^2} + \frac{6}{\rho^3}\right)e^{-\rho} + \frac{6}{\rho^3}$$

where the subscripts S and F denote, respectively, the singular and the formal part of the derivatives

$$\nabla_i\nabla_j J^{(1)} = \nabla_i\nabla_j J_S^{(1)} + \nabla_i\nabla_j J_F^{(1)}$$

Formulas (2.6),(2.10) and (2.14) define the correlation function of the displacement vector. Correlation functions of the strain tensor and angles of rotation can be written, in the terms of U_{ij} as follows:

$$E_{ijmn} = -\nabla_{(i}U_{j)(m,n)}, \qquad \Omega_{ij} = -{}^1/_4 e_{ipq}e_{jrs}U_{pr,qs} \qquad (2.15)$$

For the correlation function of the stress tensor we have

$$S_{ijmn} = A_{mnpq}^{ijkl}\langle\varepsilon_{kl}\rangle\langle\varepsilon_{pq}\rangle\varphi + \langle\lambda_{ijkl}\rangle\langle\lambda_{mnpq}\rangle E_{klpq} +$$
$$+ \langle\lambda_{ijkl}\rangle\langle\varepsilon_{pq}\rangle\langle\varepsilon_{kl}'\lambda_{mnpq}'\rangle + \langle\lambda_{mnpq}\rangle\langle\varepsilon_{kl}\rangle\langle\varepsilon_{pq}'\lambda_{ijkl}'\rangle \qquad (2.16)$$

Computing the mixed correlation function $\langle u_k'\lambda_{mnpq}\rangle$ we obtain

$$\langle u_k'(\mathbf{r}+\boldsymbol{\rho})\lambda_{mnpq}'(\boldsymbol{\rho})\rangle = \langle G_{l,r,s} * \lambda_{rstv}'\varepsilon_{tv}\lambda_{mnpq}'\rangle =$$
$$= A_{rstv}^{mnpq}\langle\varepsilon_{tv}\rangle\frac{1}{\langle\mu\rangle}[J_{,s}^{(1)}\delta_{kr} + \varkappa J_{,krs}^{(2)}] \qquad (2.17)$$

Thus (2.16) can be written as

$$S_{ijmn} = A_{mnpq}^{ijkl}\langle\varepsilon_{kl}\rangle\langle\varepsilon_{pq}\rangle\varphi + \langle\lambda_{ijkl}\rangle\langle\lambda_{mnpq}\rangle E_{klpq} +$$
$$+ A_{rstv}^{mnpq}\langle\lambda_{ijkl}\rangle\langle\varepsilon_{pq}\rangle\langle\varepsilon_{tv}\rangle I_{ls}^{kr} + A_{rstv}^{ijkl}\langle\lambda_{mnpq}\rangle\langle\varepsilon_{kl}\rangle\langle\varepsilon_{tv}\rangle I_{qs}^{pr} \qquad (2.18)$$

$$I_{ls}^{hr} \equiv \frac{1}{\langle\mu\rangle}(\delta_{kr}J_{,ls}^{(1)} + \varkappa J_{,hrls}^{(2)}) \qquad (2.19)$$

Expressions (2.6),(2.15) and (2.18) can be simplified using the following relations valid for a quasi-isotropic medium

$$A_{klrs}^{ijpq}\langle\varepsilon_{pq}\rangle\langle\varepsilon_{rs}\rangle = F_1\delta_{ij}\delta_{kl} + F_2 D_{ijkl} \qquad (2.20)$$

$$F_1 = {}^1/_9 A_{klqq}^{ijpp}\langle\varepsilon_{ij}\rangle\langle\varepsilon_{kl}\rangle, \qquad F_2 = {}^1/_{20} D_{pqrs}A_{klrs}^{ijpq}\langle\varepsilon_{ij}\rangle\langle\varepsilon_{kl}\rangle \qquad (2.21)$$

where the contractions of the tensor A_{rstv}^{hlpq} are given by (1.10),(1.18) and (1.22).

Inserting (2.20) and (2.21) into (2.6) we obtain

$$U_{ij} = (F_1 + {}^1/_3 F_2)I_{jll}^{ikk} + F_2 I_{jkl}^{ikl} \qquad (2.22)$$

The explicit form of the correlation functions of the strain tensor and rotation vector can be obtained by putting (2.22) into (2.15)

$$E_{ijkl} = \left[\frac{F_1 + {}^4/_3 F_2}{\langle\lambda+2\mu\rangle^2} - \frac{F_2}{\langle\mu\rangle^2}\right]J_{,ijkl}^{(2)} - \frac{F_2}{\langle\mu\rangle^2}\nabla_{(i}\delta_{j)(k}J_{,l)}^{(1)} \qquad (2.23)$$

$$\Omega_{ij} = \frac{F_2}{4\langle\mu\rangle^2}(\varphi\delta_{ij} + J_{,ij}^{(1)}) \qquad (2.24)$$

Inserting (2.19),(2.20) and (2.23) into (2.18), we obtain the correlation function of the stress tensor

$$S_{ijkl} = 2[(\eta\delta_{ij}\delta_{kl} + F_2\delta_{i(k}\delta_{l)j})\varphi + \eta(J_{,ij}^{(1)}\delta_{kl} + J_{,kl}^{(1)}\delta_{ij}) +$$
$$+ 2\zeta J_{,ijkl}^{(2)} + 2F_2\nabla_{(i}\delta_{j)(k}J_{,l)}^{(1)}] \qquad (2.25)$$

$$\eta \equiv \frac{(\langle K - {}^2/_3\mu\rangle^2 - {}^4/_3\langle\mu\rangle^2)F_2 + 2\langle\mu\rangle^2 F_1}{\langle\lambda+2\mu\rangle^2}, \qquad \zeta \equiv \frac{(F_1 + {}^1/_3 F_2)\langle\mu\rangle^2 + F_2\langle\lambda+\mu\rangle^2}{\langle\lambda+2\mu\rangle^2}$$

3. Let us write the strain tensor as a sum of its volume and shear components

$$\varepsilon_{ij} = {}^1/_3\varepsilon\delta_{ij} + {}^1/_2 D_{ijkl}\varepsilon_{kl} \qquad (3.1)$$

Then by (1.10) and (2.21) we shall have, for a polycrystal,

$$F_1 = \frac{1}{90} A_1 \langle e \rangle^2, \qquad F_2 = \frac{1}{90} A_1 \langle e \rangle^2 + \frac{1}{300} A_2 \langle e \rangle^2 \qquad (3.2)$$

$$\langle e \rangle^2 \equiv D_{ijkl} \langle \varepsilon_{ij} \rangle \langle \varepsilon_{kl} \rangle \qquad (3.3)$$

and for a mechnical mixture of isotropic components with (1.22) taken into account,

$$F_1 = D_K \langle \varepsilon \rangle^2, \qquad F_2 = \frac{1}{5} D_\mu \langle \varepsilon \rangle^2 \qquad (3.4)$$

From (3.2) and (3.4) it follows that we can obtain the correlation functions for arbitrary strains of an inhomogeneous material by simple summation of the correlation functions for the volume and pure shear strain. With this in mind, we can write the correlation tensors of strains and angles of rotation, as

$$E_{ijkl} = \varepsilon_{ijkl}^\varepsilon \langle \varepsilon \rangle^2 + \varepsilon_{ijkl}^e \langle e \rangle^2, \qquad \Omega_{ij} = \omega_{ij}^\varepsilon \langle \varepsilon \rangle^2 + \omega_{ij}^e \langle e \rangle^2 \qquad (3.5)$$

By (2.23) and (2.24), the coefficients ε_{ijkl} and ω_{ij} are

$$\varepsilon_{ijkl}^\varepsilon = \frac{A_1}{90 \langle \mu \rangle^2} \left[\left(\frac{4 \langle \mu \rangle^2}{3 \langle \lambda + 2\mu \rangle^2} - 1 \right) J_{,\,ijkl}^{(2)} - \nabla_{(i} \delta_{j)\,(k} J_{,\,l)}^{(1)} \right] \qquad (3.6)$$

$$\varepsilon_{ijkl}^e = \frac{3A_2}{10 A_1} \varepsilon_{ijkl}^\varepsilon + \frac{A_1}{90 \langle \lambda + 2\mu \rangle^2} J_{,\,ijkl}^{(2)}$$

$$\omega_{ij}^\varepsilon = \frac{A_1}{360 \langle \mu \rangle^2} (\varphi \delta_{ij} + J_{,\,ij}^{(1)}), \qquad \omega_{ij}^e = \frac{3A_2}{10 A_1} \omega_{ij}^\varepsilon \qquad (3.7)$$

for the polycrystals, and

$$\varepsilon_{ijkl}^\varepsilon = \frac{D_K}{\langle \lambda + 2\mu \rangle^2} J_{,\,ijkl}^{(2)}$$

$$\varepsilon_{ijkl}^e = \frac{D_\mu}{5 \langle \mu \rangle^2} \left[\left(\frac{4 \langle \mu \rangle^2}{3 \langle \lambda + 2\mu \rangle^2} - 1 \right) J_{,\,ijkl}^{(2)} - \nabla_{(i} \delta_{j)\,(k} J_{,\,l)}^{(1)} \right] \qquad (3.8)$$

$$\omega_{ij}^\varepsilon = 0, \qquad \omega_{ij}^e = \frac{D_\mu}{20 \langle \mu \rangle^2} (\varphi \delta_{ij} + J_{,\,ij}^{(1)}) \qquad (3.9)$$

for the mechanical mixtures.

The condition $\omega_{ij}^\varepsilon = 0$ means, that the volume strain of the mechanical mixtures of isotropic components is not accompanied by the rotation of the grains. This is also true for the volume strain of the polycrystals possessing cubic structure for which, in this case, the elastic field is homogeneous. For the polycrystals possessing lower order symmetry, the volume strains are accompanied by the random rotations of the crystallites.

We can obtain expressions analogous to (3.6) and (3.8) for the correlation function of the stress tensor S_{ijkl}, by separating the quantity F_i into its deviation and volume components and writing analogous relations for η and ζ.

The scalar function $E_{iikk}(\mathbf{r})$ gives the correlation between the random components of volume strains at the points $\mathbf{r} + \rho$ and ρ, while the contraction of $E_{ijkl}(\mathbf{r})$ with the deviation tensor yields the correlations for the shear strains.

Corresponding expressions have the form

$$E_{iikk}(\mathbf{r}) = \frac{F_1 + \frac{4}{3} F_2}{\langle \lambda + 2\mu \rangle^2} \varphi(r), \qquad D_{ijkl} E_{ijkl}(\mathbf{r}) = \frac{4}{3} E_{iikk}(\mathbf{r}) + \frac{2F_2}{\langle \mu \rangle^2} \varphi(r) \qquad (3.10)$$

which yields

$$E_{iikk}(\mathbf{r}) = \frac{\varphi(r)}{1350 \langle \lambda + 2\mu \rangle^2} [20 A_1 \langle \varepsilon \rangle^2 + 3(5A_1 + 2A_2) \langle e \rangle^2] \qquad (3.11)$$

$$D_{ijkl}E_{ijkl}(\mathbf{r}) = \frac{\varphi(r)}{4050\langle\lambda + 2\mu\rangle^2}[10pA_1\langle e\rangle^2 + 3(20A_1 + pA_2)\langle e\rangle^2] \quad (3.12)$$

$$p \equiv 8 + 9(2 + \langle\lambda\rangle/\langle\mu\rangle)^2$$

for the polycrystal and

$$E_{iikk}(\mathbf{r}) = \frac{\varphi(r)}{15\langle\lambda + 2\mu\rangle^2}(15\ D_K\langle e\rangle^2 + 4D_\mu\langle e\rangle^2) \quad (3.13)$$

$$D_{ijkl}E_{ijkl}(\mathbf{r}) = \frac{2\varphi(r)}{45\langle\lambda + 2\mu\rangle^2}(30D_K\langle e\rangle^2 + pD_\mu\langle e\rangle^2) \quad (3.14)$$

for the mechanical mixtures.

From $(3.10) - (3.14)$ it follows, that the radial dependence of both, contractions of the correlation tensors of the strain field and contractions of the correlation function of the elastic moduli tensor, has the same character. Generally speaking, the correlations between the random components of the volume or shear strain exist in both these cases of macro-strain. In some cases however, they may become simpler. Thus for the cubic polycrystals $A_1 = 0$, therefore the random component of the elastic field is absent in the volume strains.

In the case of mechanical mixtures, $\langle\varepsilon\rangle^2$ and $\langle e\rangle^2$ are preceded by the autocorrelation coefficients of the corresponding elastic moduli. The latter means, that the necessary condition for the random component of shear strains to appear when $\langle\varepsilon\rangle = 0$ is, that the moduli of the shear components assume different values. The random component of the shear strain may also appear when the shear moduli coincide, but then we must have $\langle\varepsilon\rangle \neq 0$. Analogous argument holds for the random component of the volume strains.

Correlations between the rotations of individual grains are given by the function Ω_{ij}, ita contraction being given by

$$\Omega_{ii}(\mathbf{r}) = \frac{F_2}{2\langle\mu\rangle^2}\varphi(r) \quad (3.15)$$

Thus we obtain

$$\Omega_{ii}(\mathbf{r}) = \frac{\varphi(r)}{1800\langle\mu\rangle^2}(10A_1\langle\varepsilon\rangle^2 + 3A_2\langle e\rangle^2) \quad (3.16)$$

$$\Omega_{ii}(\mathbf{r}) = \frac{D_\mu}{10\langle\mu\rangle^2}\varphi(r)\langle e\rangle^2 \quad (3.17)$$

for the polycrystal and for the mechanical mixture respectively.

Expression Ω_{ii} can be regarded as a scalar product of random vectors taken at the points $\mathbf{r} + \boldsymbol{\rho}$ and $\boldsymbol{\rho}$. We can easily see that the correlation function constructed on the basis of vector products of the corresponding vectors becomes zero i.e. $e_{ijk}\Omega_{jk} = 0$. This follows from the properties of the model of the quasi-isotropic space which is used here, and in which $\Omega_{ij}(\mathbf{r}) = \Omega_{ji}(\mathbf{r})$.

Similar relations can be written for the correlation function of the stress tensor. Coordinate dependence of the corresponding contractions is also given by the function $\varphi(r)$. This follows from

$$S_{iikk}(\mathbf{r}) = 2(3\eta + 2\zeta + F_2)\varphi(r), \quad D_{ijkl}S_{ijkl}(\mathbf{r}) = \tfrac{4}{3}(4\zeta + 5F_2)\varphi(r) \quad (3.18)$$

In the case of a polycrystal, contractions of the correlation tensor of the stress field have the form

$$S_{iikk}(\mathbf{r}) = \frac{\varphi(r)}{225\langle\lambda + 2\mu\rangle^2}[30\langle K\rangle^2 A_1\langle\varepsilon\rangle^2 + (40\langle\mu\rangle^2 A_1 + 9\langle K\rangle^2 A_2)\langle e\rangle^2]$$

$$(3.19)$$

$$D_{ijkl}S_{ijkl}(\mathbf{r}) = \frac{\varphi(r)}{2025 \langle \lambda + 2\mu \rangle^2} [10q A_1 \langle \varepsilon \rangle^2 + 3(40 \langle \mu \rangle^2 A_1 + q A_2) \langle e \rangle^2]$$

$$(3.20)$$

$$q \equiv 27 \langle K \rangle^2 + 48 \langle K\mu \rangle + 32 \langle \mu \rangle^2$$

while for the mechanical mixture we have, respectively

$$S_{iikk}(\mathbf{r}) = \frac{4\varphi(r)}{5 \langle \lambda + 2\mu \rangle^2} (20 \langle \mu \rangle^2 D_K \langle \varepsilon \rangle^2 + 3 \langle K \rangle^2 D_\mu \langle e \rangle^2) \qquad (3.21)$$

$$D_{ijkl}S_{ijkl}(\mathbf{r}) = \frac{4\varphi(r)}{45 \langle \lambda + 2\mu \rangle^2} (60 \langle \mu \rangle^2 D_K \langle \varepsilon \rangle^2 + q D_\mu \langle e \rangle^2) \qquad (3.22)$$

4. Let us compare the fluctuation characteristics of the stress and strain fields. We introduce the following dimensionless partial correlation functions

$$E_\varepsilon = (\varepsilon_{iikk}^\varepsilon)^{1/2}, \qquad E_e = (D_{ijkl}\varepsilon_{ijkl}^e)^{1/2} \qquad (4.1)$$

$$S_\varepsilon = \frac{1}{3 \langle K \rangle \langle \varepsilon \rangle} (S_{iikk}^\varepsilon)^{1/2}, \qquad S_e = \frac{1}{2 \langle \mu \rangle \langle e \rangle} (D_{ijkl}S_{ijkl}^e)^{1/2}$$

The subscript accompanying E and S shows, which of the components of the strain tensor (shear e or volume ε) is different from zero. Thus when $r = 0$, the function E_ε gives the respective spatial fluctuation of the volume strains appearing when strain takes place without shear, while E_e yields the relative fluctuation of the shear strains taking place when the material is strained with its mean volume kept constant.

Computing E and S for mechanical mixtures, we obtain

$$E_\varepsilon^2 = \frac{D_K \varphi(r)}{\langle \lambda + 2\mu \rangle^2}, \qquad E_e^2 = \frac{2p D_\mu \varphi(r)}{45 \langle \lambda + 2\mu \rangle^2}$$

$$S_\varepsilon^2 = \frac{16 \langle \mu \rangle^2 D_K \varphi(r)}{9 \langle K \rangle^2 \langle \lambda + 2\mu \rangle^2}, \qquad S_e^2 = \frac{q D_\mu \varphi(r)}{45 \langle \mu \rangle^2 \langle \lambda + 2\mu \rangle^2} \qquad (4.2)$$

which, in turn, yields the ratios of the dimensionless partial correlation functions of the stress and strain fields

$$\frac{S_\varepsilon}{E_\varepsilon} = \frac{4 \langle \mu \rangle}{3 \langle K \rangle}, \qquad \frac{S_e}{E_e} = \frac{1}{\langle \mu \rangle} \left(\frac{q}{2p} \right)^{1/2} \qquad (4.3)$$

From (4.3) we see that the relative spatial fluctuations of the stress and strain fields coincide only when $\langle K \rangle = {}^4/_3 \langle \mu \rangle$. For volume strains, the ratio of the dimensionless partial correlation functions yields directly the coefficient ${}^4/_3 \langle \mu \rangle \langle K \rangle^{-1}$, while in the case of shear strains we have, on inserting the explicit expressions for p and q into (4.3),

$$\frac{S_e}{E_e} = \left[1 + \frac{\langle 3K - 4\mu \rangle \langle 3K + 4\mu \rangle}{9 \langle K \rangle^2 + 24 \langle K \rangle \langle \mu \rangle + 24 \langle \mu \rangle^2} \right]^{1/2} \qquad (4.4)$$

The second critical case takes place when the relative spatial fluctuations of the elastic field occur under pure shear, and shear fluctuations under volume strains. Putting

$$\varepsilon_* = (E_{iikk}^\varepsilon)^{1/2}, \qquad e_* = (D_{ijkl}E_{ijkl}^e)^{1/2}, \qquad \sigma_* = (S_{iikk}^\varepsilon)^{1/2}, \qquad s_* = (D_{ijkl}S_{ijkl}^e)^{1/2} \quad (4.5)$$

we can find the connection between the characteristics of the random stress and strain fields

$$\sigma_* = 3 \langle K \rangle \varepsilon_*, \qquad s_* = 2 \langle \mu \rangle e_* \qquad (4.6)$$

We note that when $\mu = 0$ (limit transition of an emulsion or a mixture of polymers possessing the property of fluidity after the full relaxation of stresses), the correlation stress tensor S_{ijkl} becomes zero as expected, while the contractions of the correlation tensor of strains assume the form

$$E_{iikk} = {}^3/_4 D_{ijkl} E_{ijkl} = \langle K \rangle^{-2} D_K \langle \varepsilon \rangle^2 \, \varphi \,(r) \tag{4.7}$$

For polycrystals, (4.1) is replaced by

$$E_\varepsilon{}^2 = S_\varepsilon{}^2 = \frac{2A_1 \varphi \,(r)}{135 \,\langle \lambda + 2\mu \rangle^2}, \qquad E_e{}^2 = \frac{(20 A_1 + p A_2) \,\varphi \,(r)}{1350 \,\langle \lambda + 2\mu \rangle^2} \tag{4.8}$$

$$S_e{}^2 = \frac{(40 \,\langle \mu \rangle^2 \, A_1 + q A_2) \,\varphi \,(r)}{2700 \,\langle \mu^2 \rangle \,\langle \lambda + 2\mu \rangle^2}$$

From this we see that, unlike the case of mixtures of the isotropic components, here the relative spatial fluctuations of the volume stresses and strains coincide for all polycrystals. When the macro-strains are pure shear, the dimensionless relative correlation functions for the shear components of the elastic field will coincide, as in the case of mechanical mixtures, only when $\langle K \rangle = {}^4/_3 \,\langle \mu \rangle$. To confirm this, it is sufficient to represent $\dot{S}_e / E_e$ as

$$\frac{S_e}{E_e} = \left[1 + \frac{\langle 3K - 4\mu \rangle \,\langle 3K + 4\mu \rangle \, A_2}{2 \,\langle \mu \rangle^2 \,(20 \, A_1 + p A_2)} \right]^{1/2} \tag{4.9}$$

For the characteristics of the random volume components of the elastic field appearing during the average shear strains and of the random shear strains and stresses appearing when the polycrystal undergoes volume strain, we have, instead of (4.6),

$$\sigma_* = 3 \,\langle K \rangle \, \varepsilon_* \left[1 - \frac{5 A_1 \,\langle 3K - 4\mu \rangle \,\langle 3K + 4\mu \rangle}{9 \,\langle K \rangle^2 \,(5 A_1 + 2 A_2)} \right]^{1/2} \tag{4.10}$$

$$s_* = 2 \,\langle \mu \rangle \, e_* \left[1 - \frac{\langle 3K - 4\mu \rangle \,\langle 3K + 4\mu \rangle}{9 \,\langle K \rangle^2 + 24 \,\langle K \rangle \,\langle \mu \rangle + 24 \,\langle \mu \rangle^2} \right]^{1/2} \tag{4.11}$$

Expressions (4.10) and (4.11) reduce to (4.6) when $\langle K \rangle = {}^4/_3 \langle \mu \rangle$.

As we noted before, the coordinate relationship is the same for the contractions of the elastic field correlation tensors and for the binary correlation function of the elastic moduli tensor. This relationship however, becomes more complicated for the arbitrary components of the elastic field correlation tensors. General expression for the second and fourth rank tensors has, in the quasi-isotropic space, the form [7 and 8]

$$T_{ij} \,(\mathbf{r}) = T_1 \,(r) \, \delta_{ij} + T_2 \,(r) \, \psi_{ij} \tag{4.12}$$

$$T_{ijkl} \,(\mathbf{r}) = T_1 \,(r) \, \delta_{ij} \delta_{kl} + T_2 \,(r) \, D_{ijkl} + T_3 \,(r) \,(\delta_{ij} \psi_{kl} + \psi_{ij} \delta_{kl}) +$$

$$+ \, T_4 \,(r) \, n_{(i} \delta_{j)(k} n_{l)} + T_5 \,(r) \, \psi_{ijkl} \tag{4.13}$$

$$\psi_{ij} \equiv n_i n_j, \qquad \psi_{ijkl} \equiv n_i n_j n_k n_l, \qquad \mathbf{n} \equiv \mathbf{r} / r \tag{4.14}$$

Let us now obtain the functions $T_1 \,(r)$ and $T_2 \,(r)$ for the correlation tensor Ω_{ij} in their explicit form. Inserting (2.13) into (2.24) we obtain

$$\Omega_{ij} \,(\mathbf{r}) = \Omega_1 \,(r) \, \delta_{ij} + \Omega_2 \,(r) \, \psi_{ij} \tag{4.15}$$

$$\Omega_1 \,(r) = \frac{F^2}{4 \,\langle \mu \rangle^2} [\,\varphi \,(r) + \varphi_* \,(r)], \qquad \Omega_2 \,(r) = - \Omega_1 \,(r) - \frac{F_2}{2 \,\langle \mu \rangle^2} \varphi_* \,(r) \tag{4.16}$$

$$\varphi_* \,(r) \equiv \frac{a}{r} \left[\left(1 + \frac{2a}{r} + \frac{2a^2}{r^2} \right) \varphi \,(r) - \frac{2a^2}{r^2} \right] \tag{4.17}$$

We easily see that the contraction Ω_{ii} leads to Formula (3.15). From (4.16) and (4.17) it follows that when $r \to 0$ and $r \to \infty$, the functions Ω_1 and Ω_2 are

$$\Omega_1 \,(0) = \frac{F_2}{6 \,\langle \mu \rangle^2}, \qquad \Omega_2 \,(0) = \frac{F_2}{16 \,\langle \mu \rangle^2} r, \qquad \Omega_2 \,(\infty) \doteq - 3 \Omega_1 \,(\infty) = \frac{3 F_2 a^3}{2 \,\langle \mu \rangle^2} \frac{1}{r^3} \tag{4.18}$$

i.e. when $r = 0$, then the second term in (4.15) vanishes, while when $r \to \infty$, then Ω_i (r) decreases at the slower rate than the exponential expression defining φ (r).

Similarly we can obtain the functions T_i (r) for the fourth rank correlation tensors of the strain and stress fields $E_{ijkl}(r)$ and S_{ijkl} (r).

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Translated by L. K.